

ON THE FACET IDEAL OF AN EXPANDED SIMPLICIAL COMPLEX

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ABSTRACT. For a simplicial complex Δ , the affect of the expansion functor on combinatorial properties of Δ and algebraic properties of its Stanley-Reisner ring has been studied in some previous papers. In this paper, we consider the facet ideal $I(\Delta)$ and its Alexander dual which we denote by J_Δ to see how the expansion functor alter the algebraic properties of these ideals. It is shown that for any expansion Δ^α the ideals J_Δ and J_{Δ^α} have the same total Betti numbers and their Cohen-Macaulayness are equivalent, which implies that the regularities of the ideals $I(\Delta)$ and $I(\Delta^\alpha)$ are equal. Moreover, the projective dimensions of $I(\Delta)$ and $I(\Delta^\alpha)$ are compared. In the sequel for a graph G , some properties that are equivalent in G and its expansions are presented and for a Cohen-Macaulay (resp. sequentially Cohen-Macaulay and shellable) graph G , we give some conditions for adding or removing a vertex from G , so that the remaining graph is still Cohen-Macaulay (resp. sequentially Cohen-Macaulay and shellable).

1. Introduction

Making modifications to a simplicial complex or a monomial ideal so that they fulfill some special properties is a tool to construct new objects with some desired properties and has been considered in many research papers, see for example [1, 2, 4, 5, 9]. In [9] the expansion functor on a simplicial complex was defined and some algebraic and combinatorial properties of a simplicial complex and its expansions were compared. Generalizing the results in [9], the authors, in [11] studied the Stanley-Reisner ideal of a simplicial complex and that of its expansions and it was proved that properties like being Cohen-Macaulay, sequentially Cohen-Macaulay, Buchsbaum and k -decomposable for these ideals are equivalent.

Since any squarefree monomial I ideal may also be considered as the facet ideal of a simplicial complex, this natural question arises that how does the expansion functor on a simplicial complex affect algebraic properties of the facet ideal of it. In this paper we consider the facet ideal of a simplicial complex Δ , and its Alexander dual J_Δ via the expansion functor on Δ .

The paper proceeds as follows. In the first section, we recall some preliminaries which are needed in the sequel. Section 2 is devoted to the study of the facet ideal of an expanded complex and its Alexander dual. One of the main results is the following theorem.

Proposition 1.1. (*Proposition 3.4*) Let Δ be a simplicial complex, $\alpha \in \mathbb{N}^n$ and J_Δ denotes the Alexander dual of the facet ideal $I(\Delta)$. Then

- (i) $\beta_i(S/J_\Delta) = \beta_i(S^\alpha/J_{\Delta^\alpha})$;
- (ii) S/J_Δ is Cohen-Macaulay if and only if $S^\alpha/J_{\Delta^\alpha}$ is Cohen-Macaulay.

This implies that $\text{reg}(I(\Delta)) = \text{reg}(I(\Delta^\alpha))$. Moreover, $I(\Delta)$ has a linear resolution if and only if $I(\Delta^\alpha)$ has a linear resolution. In Proposition 3.8, we prove that if the facet ideal $I(\Delta)$ has linear quotients and $\alpha = (s, s, \dots, s)$, then $\text{pd}(I(\Delta^\alpha)) \leq \text{pd}(I(\Delta))s + (d+1)(s-1)$, where $d = \dim(\Delta)$. Moreover, if Δ is pure, then

$$\text{pd}(I(\Delta^\alpha)) = \text{pd}(I(\Delta))s + (d+1)(s-1).$$

In Section 3, we consider the case that $\Delta = G$ is a graph. In Theorem 4.1, we find some properties that are equivalent in G and its expansions. We show that for a Cohen-Macaulay (resp. sequentially Cohen-Macaulay and shellable) graph G and a vertex $x \in V(G)$, if we add a new vertex x' to G and connect it to x and all of its neighbours, then the new graph is again Cohen-Macaulay (resp. sequentially Cohen-Macaulay and shellable). Also we give a condition so that by removing a vertex x from G , the graph $G \setminus x$ is still Cohen-Macaulay (resp. sequentially Cohen-Macaulay and shellable) (see Corollaries 4.3, 4.4).

2. PRELIMINARIES

Throughout this paper, we assume that Δ is a simplicial complex on the vertex set $X = \{x_1, \dots, x_n\}$, K is a field and $S = K[X]$ is a polynomial ring. The set of facets (maximal faces) of Δ is denoted by $\mathcal{F}(\Delta)$ and if $\mathcal{F}(\Delta) = \{F_1, \dots, F_r\}$, we write $\Delta = \langle F_1, \dots, F_r \rangle$. For a monomial ideal I of S , the set of minimal generators of I is denoted by $\mathcal{G}(I)$. For $\alpha = (s_1, \dots, s_n) \in \mathbb{N}^n$, we set $X^\alpha = \{x_{11}, \dots, x_{1s_1}, \dots, x_{n1}, \dots, x_{ns_n}\}$ and $S^\alpha = K[X^\alpha]$.

The concept of expansion of a simplicial complex was defined in [9] as follows.

Definition 2.1. Let Δ be a simplicial complex on X , $\alpha = (s_1, \dots, s_n) \in \mathbb{N}^n$ and $F = \{x_{i_1}, \dots, x_{i_r}\}$ be a facet of Δ . The **expansion** of the simplex $\langle F \rangle$ with respect to α is denoted by $\langle F \rangle^\alpha$ and is defined as a simplicial complex on the vertex set $\{x_{i_l t_l} : 1 \leq l \leq r, 1 \leq t_l \leq s_{i_l}\}$ with facets

$$\{\{x_{i_1 j_1}, \dots, x_{i_r j_r}\} : 1 \leq j_m \leq s_{i_m}\}.$$

The expansion of Δ with respect to α is defined as

$$\Delta^\alpha = \bigcup_{F \in \Delta} \langle F \rangle^\alpha.$$

A simplicial complex obtained by an expansion, is called an expanded complex.

Definition 2.2. A monomial ideal I in the ring S has **linear quotients** if there exists an ordering f_1, \dots, f_m on the minimal generators of I such that the colon ideal $(f_1, \dots, f_{i-1}) : (f_i)$ is generated by a subset of $\{x_1, \dots, x_n\}$ for all $2 \leq i \leq m$. We show this ordering by $f_1 < \dots < f_m$ and we call it an order of linear quotients on $\mathcal{G}(I)$.

Let I be a monomial ideal which has linear quotients and $f_1 < \dots < f_m$ be an order of linear quotients on the minimal generators of I . For any $1 \leq i \leq m$, $\text{set}_I(f_i)$ is defined as

$$\text{set}_I(f_i) = \{x_k : x_k \in (f_1, \dots, f_{i-1}) : (f_i)\}.$$

For a \mathbb{Z} -graded S -module M , the **Castelnuovo-Mumford regularity** (or briefly regularity) of M is defined as

$$\text{reg}(M) = \max\{j - i : \beta_{i,j}(M) \neq 0\},$$

and the **projective dimension** of M is defined as

$$\text{pd}(M) = \max\{i : \beta_{i,j}(M) \neq 0 \text{ for some } j\},$$

where $\beta_{i,j}(M)$ is the (i, j) th graded Betti number of M .

For a simplicial complex Δ with the vertex set X , the **Alexander dual simplicial complex** associated to Δ is defined as

$$\Delta^\vee = \{X \setminus F : F \notin \mathcal{F}(\Delta)\}.$$

For a squarefree monomial ideal $I = (x_{11} \cdots x_{1n_1}, \dots, x_{t1} \cdots x_{tn_t})$, the **Alexander dual ideal** of I , denoted by I^\vee , is defined as

$$I^\vee := (x_{11}, \dots, x_{1n_1}) \cap \cdots \cap (x_{t1}, \dots, x_{tn_t}).$$

For a subset $C \subseteq X$, by x^C we mean the monomial $\prod_{x \in C} x$. One can see that

$$(I_\Delta)^\vee = (x^{F^c} : F \in \mathcal{F}(\Delta)),$$

where I_Δ is the Stanley-Reisner ideal associated to Δ and $F^c = X \setminus F$. Moreover, $(I_\Delta)^\vee = I_{\Delta^\vee}$.

A simplicial complex Δ is called Cohen-Macaulay (resp. sequentially Cohen-Macaulay, Buchsbaum and Gorenstein), if its the Stanley Reisner ring $K[\Delta] = S/I_\Delta$ is Cohen-Macaulay (resp. sequentially Cohen-Macaulay, Buchsbaum and Gorenstein). For a graph G , with the vertex set $V(G)$ and the edge set $E(G)$, the **independence complex** of G is defined as

$$\Delta_G = \{F \subseteq V(G) : e \not\subseteq F, \forall e \in E(G)\}.$$

The graph G is called Cohen-Macaulay (resp. sequentially Cohen-Macaulay and shellable) if Δ_G is Cohen-Macaulay (resp. sequentially Cohen-Macaulay and shellable).

For a simplicial complex Δ , the facet ideal of Δ is defined as $I(\Delta) = (x^F : F \in \mathcal{F}(\Delta))$. Also the complement of Δ is the simplicial complex $\Delta^c = \langle F^c : F \in \mathcal{F}(\Delta) \rangle$. In fact $I(\Delta^c) = I_{\Delta^\vee}$.

3. ON THE FACET IDEAL OF AN EXPANDED COMPLEX

This section is devoted to the study of facet ideal of an expanded complex and its Alexander dual to see how their algebraic properties change via the expansion functor. Set $J_\Delta := I_{\Delta^c}$. Then one can see that $(J_\Delta)^\vee = I(\Delta)$ and $J_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_F$, where $P_F = (x_i : x_i \in F)$.

For any $1 \leq i \leq n$, let $\delta_i = (a_1, \dots, a_n) \in \mathbb{N}^n$, where the components are defined as $a_j = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i. \end{cases}$ Also let $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{N}^n$.

Lemma 3.1. *Let Δ be a simplicial complex on X , $\beta = (k_1, \dots, k_n) \in \mathbb{N}^n$ and $\alpha = \beta + \delta_i$. Then $(\Delta^\beta)^{\mathbf{1}+\delta_{ik_i}} \cong \Delta^\alpha$.*

Proof. Note that

$$X^\alpha = \{x_{11}, \dots, x_{1k_1}, \dots, x_{i1}, \dots, x_{ik_i}, x_{i(k_i+1)}, \dots, x_{n1}, \dots, x_{nk_n}\}$$

and

$$(X^\beta)^{\mathbf{1}+\delta_{ik_i}} = \{x_{111}, \dots, x_{1k_11}, \dots, x_{i11}, \dots, x_{ik_i1}, x_{ik_i2}, \dots, x_{n11}, \dots, x_{nk_n1}\}.$$

Define $\varphi : (X^\beta)^{\mathbf{1}+\delta_{ik_i}} \rightarrow X^\alpha$ given by

$$\varphi(x_{rst}) = \begin{cases} x_{rs} & t = 1 \\ x_{i(k_i+1)} & t = 2. \end{cases}$$

Then φ induces the simplicial map

$$\begin{aligned} \theta : (\Delta^\beta)^{\mathbf{1}+\delta_{ik_i}} &\rightarrow \Delta^\alpha. \\ F &\mapsto \varphi(F) \end{aligned}$$

which is an isomorphism. \square

The following lemma explains the generators of J_{Δ^α} in terms of the generators of J_Δ .

Lemma 3.2. *Let Δ be a simplicial complex and let $\alpha = (k_1, \dots, k_n) \in \mathbb{N}^n$. Then*

$$J_{\Delta^\alpha} = (\{\prod_{j=1}^{k_{i_1}} x_{i_1j} \cdots \prod_{j=1}^{k_{i_r}} x_{i_rj} : x_{i_1} \cdots x_{i_r} \in J_\Delta\}).$$

Proof. It suffices to show “ \subseteq ”. We use induction on α .

Let $\alpha = \mathbf{1} + \delta_1$. When $\Delta = \langle \{x_1, \dots, x_n\} \rangle$ then $J_{\Delta^\alpha} = \bigcap_{F \in \mathcal{F}(\Delta^\alpha)} P_F = (x_{11}x_{12}, x_2, \dots, x_n)$ and the assertion holds. Let $\Delta = \langle F_1, \dots, F_s \rangle$ and $x_1 \in F_i$ for $i = 1, \dots, r$, and $x_1 \notin F_i$ for $i = r+1, \dots, s$. Then

$$\begin{aligned} J_{\Delta^\alpha} &= [\bigcap_{F \in \mathcal{F}(\langle F_1 \rangle^\alpha)} P_F] \cap \cdots \cap [\bigcap_{F \in \mathcal{F}(\langle F_s \rangle^\alpha)} P_F] \\ &= [(x_{11}x_{12}) + P_{F_1 \setminus \{x_1\}}] \cap \cdots \cap [(x_{11}x_{12}) + P_{F_r \setminus \{x_1\}}] \cap [\bigcap_{i=r+1}^s P_{F_i}] \\ &= [(x_{11}x_{12}) + \bigcap_{i=1}^r P_{F_i \setminus \{x_1\}}] \cap [\bigcap_{i=r+1}^s P_{F_i}] \\ &= (x_{i_11} \cdots x_{i_t1} : x_{i_1} \cdots x_{i_t} \in J_\Delta, i_l \neq 1 \text{ for all } l) + \\ &\quad (x_{11}x_{12}x_{i_11} \cdots x_{i_{t'}1} : x_1x_{i_1} \cdots x_{i_{t'}} \in J_\Delta). \end{aligned}$$

Suppose that $\alpha = (k_1, \dots, k_n) \in \mathbb{N}^n$ is arbitrary with $k_1 > 1$ and let $\alpha = \beta + \delta_1$. By the induction hypothesis,

$$J_{\Delta^\beta} = (\prod_{j=1}^{k_{i_1}} x_{i_1 j} \cdots \prod_{j=1}^{k_{i_t}} x_{i_t j} : x_{i_1} \cdots x_{i_t} \in J_\Delta, i_l \neq 1 \text{ for all } l) + \\ (\prod_{j=1}^{k_1-1} x_{1j} \prod_{j=1}^{k_{i_1}} x_{i_1 j} \cdots \prod_{j=1}^{k_{i_{t'}}} x_{i_{t'} j} : x_1 x_{i_1} \cdots x_{i_{t'}} \in J_\Delta),$$

and so

$$J_{(\Delta^\beta)^{1+\delta_1 k_1}} = (\prod_{j=1}^{k_{i_1}} x_{i_1 j 1} \cdots \prod_{j=1}^{k_{i_t}} x_{i_t j 1} : x_{i_1} \cdots x_{i_t} \in J_\Delta, i_l \neq 1 \text{ for all } l) + \\ (x_1 (k_1-1)_2 \prod_{j=1}^{k_1-1} x_{1j 1} \prod_{j=1}^{k_{i_1}} x_{i_1 j 1} \cdots \prod_{j=1}^{k_{i_{t'}}} x_{i_{t'} j 1} : x_1 x_{i_1} \cdots x_{i_{t'}} \in J_\Delta).$$

It follows from Lemma 3.1 that

$$J_{\Delta^\alpha} = (\{\prod_{j=1}^{k_{i_1}} x_{i_1 j} \cdots \prod_{j=1}^{k_{i_r}} x_{i_r j} : x_{i_1} \cdots x_{i_r} \in J_\Delta\}).$$

□

To prove Proposition 3.4, we use the following Proposition.

Proposition 3.3. ([7, Proposition 1]) *Let R be a Noetherian local ring containing a field K , and u_1, \dots, u_n be an R -sequence. Then the natural map*

$$\varphi : S = K[x_1, \dots, x_n] \rightarrow R \\ x_i \mapsto u_i$$

of K -algebras is injective and R is a flat S -module.

Proposition 3.4. *Let Δ be a simplicial complex and $\alpha = (k_1, \dots, k_n) \in \mathbb{N}^n$. Then*

- (i) $\beta_i(S/J_\Delta) = \beta_i(S^\alpha/J_{\Delta^\alpha})$;
- (ii) S/J_Δ is Cohen-Macaulay if and only if $S^\alpha/J_{\Delta^\alpha}$ is Cohen-Macaulay.

Proof. Define $\varphi : S \rightarrow S^\alpha$ given by $\varphi(x_i) = \prod_{j=1}^{k_i} x_{ij}$. Since $\prod_{j=1}^{k_1} x_{1j}, \dots, \prod_{j=1}^{k_n} x_{nj}$ is an S^α -regular sequence, it follows from Proposition 3.3 that S^α is a flat S -module. Now, by [3, Theorem 2.1.7], (ii) is concluded. Also, if F_\bullet is a minimal free resolution of S^α/J_Δ over S , then it follows that $F_\bullet \otimes_S S^\alpha$ is a minimal free resolution of $S^\alpha/J_{\Delta^\alpha}$ over S^α . Therefore we obtain (i). □

The following corollary shows that the regularity of the facet ideal does not change under the expansion functor.

Corollary 3.5. *Let Δ be a simplicial complex and let $\alpha \in \mathbb{N}^n$. Then $\text{reg}(I(\Delta)) = \text{reg}(I(\Delta^\alpha))$. Moreover, $I(\Delta)$ has a linear resolution if and only if $I(\Delta^\alpha)$ has a linear resolution.*

Proof. It is a consequence of Proposition 3.4, [15, Theorem 2.1, Corollary 1.6]. □

Remark 3.6. Bayati and Herzog in [1] defined the expansion functor in the category of finitely generated multigraded S -modules. They showed that a finitely generated multigraded S -module has a linear resolution if and only if its expansion does, too (c.f. [1, Corollary 4.3]). A special case of their result is the second part of Corollary 3.5. Then in [13], the authors studied the expansion of monomial ideals in the concept of Bayati and Herzog. They showed that a monomial ideal has linear quotients if and only if its expansion does (c.f. [13, Theorem 1.7]). Also, a monomial ideal is weakly polymatroidal if and only if its expansion is (c.f. [13, Theorem 1.4]). As a consequence of these results we have:

If Δ is a simplicial complex on $[n]$ and $\alpha \in \mathbb{N}^n$, then

- $I(\Delta)$ has linear quotients if and only if $I(\Delta^\alpha)$ has linear quotients;
- $I(\Delta)$ is weakly polymatroidal if and only if $I(\Delta^\alpha)$ is weakly polymatroidal.

We use the following theorem to compare the projective dimension of a facet ideal $I(\Delta)$ with linear quotients with the projective dimension of $I(\Delta^\alpha)$.

Theorem 3.7. ([14, Corollary 2.7]) *Let I be a monomial ideal with linear quotients with the ordering $f_1 < \dots < f_m$ on the minimal generators of I . Then*

$$\beta_{i,j}(I) = \sum_{\deg(f_t)=j-i} \binom{|\text{set}_I(f_t)|}{i}.$$

Proposition 3.8. *If $I(\Delta)$ has linear quotients, $\alpha = (s, s, \dots, s)$ and $d = \dim(\Delta)$, then $\text{pd}(I(\Delta^\alpha)) \leq \text{pd}(I(\Delta))s + (d+1)(s-1)$. Moreover, if Δ is pure, then*

$$\text{pd}(I(\Delta^\alpha)) = \text{pd}(I(\Delta))s + (d+1)(s-1).$$

Proof. Let $I(\Delta)$ has linear quotients. In view of the proof of [13, Theorem 1.7], consider an order on the minimal generators of $I(\Delta^\alpha)$ as follows.

Fix an order of linear quotients for $I(\Delta)$. For two facets $F, F' \in \Delta^\alpha$, if $\overline{F} = \overline{F'} = \{x_{i_1}, \dots, x_{i_k}\}$, $F = \{x_{i_1 r_1}, \dots, x_{i_k r_k}\}$ and $F' = \{x_{i_1 r'_1}, \dots, x_{i_k r'_k}\}$, where $i_1 < \dots < i_k$, set $x^F < x^{F'}$ if and only if $(r_1, \dots, r_k) <_{lex} (r'_1, \dots, r'_k)$. Otherwise set $x^F < x^{F'}$ if and only if $x^{\overline{F}} < x^{\overline{F'}}$ in the order of linear quotients for $I(\Delta)$.

$I(\Delta^\alpha)$ has linear quotients with respect to above order. In the light of Theorem 3.7, with this ordering, we have $\text{pd}(I(\Delta^\alpha)) = \max\{|\text{set}_{I(\Delta^\alpha)}(x^F)| \mid F \in \mathcal{F}(\Delta^\alpha)\}$. One can see that for a minimal generator $x^F \in I(\Delta^\alpha)$,

$$\text{set}_{I(\Delta^\alpha)}(x^F) = \{x_{it_i} : x_i \in \text{set}_{I(\Delta)}(x^{\overline{F}}), 1 \leq t_i \leq s\} \cup \{x_{it_i} : x_{ir_i} \in F, t_i < r_i\}.$$

Thus for any $F \in \mathcal{F}(\Delta^\alpha)$,

$$|\text{set}_{I(\Delta^\alpha)}(x^F)| \leq |\text{set}_{I(\Delta)}(x^{\overline{F}})|s + |F|(s-1) \leq \text{pd}(I(\Delta))s + (d+1)(s-1).$$

Therefore $\text{pd}(I(\Delta^\alpha)) \leq \text{pd}(I(\Delta))s + (d+1)(s-1)$.

Now, assume that Δ is pure of dimension d and let $\text{pd}(I(\Delta)) = |\text{set}_{I(\Delta)}(x^F)|$ for some $F \in \mathcal{F}(\Delta)$. Let $F = \{x_{i_1}, \dots, x_{i_t}\}$. Then for $F' = \{x_{i_1s}, \dots, x_{i_ts}\}$ one has

$$|\text{set}_{I(\Delta^\alpha)}(x^{F'})| = |\text{set}_{I(\Delta)}(x^F)|s + |F|(s-1) = \text{pd}(I(\Delta))s + (d+1)(s-1),$$

noting the fact that $\{x_{it_i} : x_i \in \text{set}_{I(\Delta)}(x^F), 1 \leq t_i \leq s\} \cap \{x_{it_i} | x_{ir_i} \in F', t_i < s\} = \emptyset$. \square

4. THE EXPANSION OF GRAPHS

In this section, we consider the case when $\Delta = G$ is a graph and investigate some properties that are equivalent in G and its expansion. We state conditions on a graph G so that by adding a vertex to G or removing a vertex from G , some properties of G like Cohen-Macaulayness are preserved.

For a graph G and a vertex $x \in V(G)$, let $N_G(x) = \{y \in V(G) : \{x, y\} \in E(G)\}$ and $N_G[x] = N_G(x) \cup \{x\}$.

Let G be a simple graph with the vertex set $X = \{x_1, \dots, x_n\}$ and let $\alpha = (k_1, \dots, k_n) \in \mathbb{N}^n$. The *expansion of G with respect to α* is denoted by G^α and it is a simple graph with the vertex set X^α and the edge set

$$E(G^\alpha) = \{\{x_{ir}, x_{js}\} : \{x_i, x_j\} \in E(G), 1 \leq r \leq k_i, 1 \leq s \leq k_j\}.$$

In the following, the notion of a co-chordal (resp. co-shellable and co-Cohen-Macaulay) graph implies to a simple graph with chordal (resp. shellable and Cohen-Macaulay) complement.

Theorem 4.1. *Let G be a simple graph and $\alpha \in \mathbb{N}^n$.*

- (i) *G is co-chordal if and only if G^α is;*
- (ii) *G is co-shellable if and only if G^α is;*
- (iii) *G is co-Cohen-Macaulay if and only if G^α is;*
- (iv) *$(\Delta_G)^\vee$ is vertex decomposable if and only if $(\Delta_{G^\alpha})^\vee$ is vertex decomposable.*

Proof. (i) In view of Corollary 3.5, the edge ideal of a simple graph G has a linear resolution if and only if the edge ideal of its expansion has a linear resolution. Combining this with Fröberg's result on edge ideals with a linear resolution (see [6, Theorem 1]), we get the assertion.

(ii), (iii) Considering the equalities $\Delta_{G^c} = \Delta(G)$ and $\Delta(G^\alpha) = \Delta(G)^\alpha$, the result follows from [11, Theorem 2.4, Corollary 2.15].

(iv) follows from (i) and [10, Corollary 3.8]. \square

Remark 4.2. There is another notion for the expansion of a graph G in the literature, which we denote it here by \widehat{G}^α , to avoid the confusion with the above concept. For $\alpha = (k_1, \dots, k_n) \in \mathbb{N}^n$, \widehat{G}^α is a simple graph with the vertex set X^α and the edge set

$$E(\widehat{G}^\alpha) = \{\{x_{ir}, x_{js}\} : \{x_i, x_j\} \in E(G), 1 \leq r \leq k_i, 1 \leq s \leq k_j\} \cup \{\{x_{ir}, x_{is}\} : 1 \leq i \leq n, r \neq s\}.$$

It is easy to see that $\Delta_{\widehat{G}^\alpha} = (\Delta_G)^\alpha$ (see for example [9, Remark 2.4]).

In view of the above remark, we conclude the following assertions.

Corollary 4.3. *Let G be a graph, $x \in V(G)$ and G' be the graph obtained from G by adding a new vertex x' and connecting it to all vertices in $N_G[x]$. If G is Cohen-Macaulay (resp. sequentially Cohen-Macaulay and shellable), then G' is Cohen-Macaulay (resp. sequentially Cohen-Macaulay and shellable).*

Proof. Note that G' is an expansion of G in the sense of Remark 4.2. Thus $\Delta_{G'}$ is an expansion of Δ_G and hence [11, Theorem 2.4, Corollaries 2.8, 2.15] imply the result. \square

Corollary 4.4. *Let G be a graph and $x, y \in V(G)$ be distinct vertices such that $N_G[x] = N_G[y]$. If G is Cohen-Macaulay (resp. sequentially Cohen-Macaulay and shellable), then $G \setminus x$ is Cohen-Macaulay (resp. sequentially Cohen-Macaulay and shellable).*

Proof. It is easy to see that G is an expansion of $G \setminus x$ in the sense of Remark 4.2. Thus Δ_G is an expansion of $\Delta_{G \setminus x}$. \square

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